

On the direct sum of two bounded linear operators and subspace-hypercyclicity

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Abstract

In this paper, we study the relation between subspace-hypercyclicity and the direct sum of two operators. In particular, we show that if the direct sum of two operators is subspace-hypercyclic, then both operators are subspace-hypercyclic; however, the converse is true for a stronger property than subspace-hypercyclicity. Moreover, we prove that if an operator T satisfies subspace-hypercyclic criterion, then $T \oplus T$ is subspace-hypercyclic. However, we show that the converse is true under certain conditions.

keywords: Hypercyclic operators, Direct sums

MSC 2010: 47A16

1 Introduction

A bounded linear operator T on a separable Banach space \mathcal{X} is hypercyclic if there is a vector $x \in \mathcal{X}$ such that $\text{Orb}(T, x) = \{T^n x : n \geq 0\}$ is dense in \mathcal{X} , such a vector x is called hypercyclic for T . The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in 1969 [11]. He showed that, if B is the backward shift on $\ell^p(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$.

In 2011, Madore and Martínez-Avendaño [9] considered the density of the orbit in a non-trivial subspace instead of the whole space, such a concept is called subspace-hypercyclicity. An operator is subspace-hypercyclic for a subspace \mathcal{M} of \mathcal{X} (or \mathcal{M} -hypercyclic, for short) if there exists a vector $x \in \mathcal{X}$ whose orbit is dense in \mathcal{M} . Also, Madore and Martínez-Avendaño [9] defined subspace-transitive operators and showed that every subspace-transitive is subspace-hypercyclic. Talebi and Moosapoor [13] defined subspace-mixing concept which is a stronger property than subspace-transitive. For more information on subspace-hypercyclicity, the reader may refer to [8, 10, 1].

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In 1982, Kitai [7] showed that if $T_1 \oplus T_2$ is hypercyclic, then T_1 and T_2 are hypercyclic. For subspace-hypercyclicity, Madore and Martínez-Avendaño showed that there exists an operator T on a Banach space \mathcal{X} such that $T \oplus I$ is subspace-hypercyclic operator for the subspace $\mathcal{X} \oplus \{0\}$ [9, Example 2.2]. However, it is clear that the identity operator I cannot be subspace-hypercyclic for any nontrivial subspace. Therefore, Kitai's result cannot be extended to subspaces. On the other hand, for nontrivial subspaces and operators, we have the following question:

Question 1. If the direct sum of two operators is subspace-hypercyclic, are both operators subspace-hypercyclic?

De la Rosa and Read [6] showed that the converse of Kitai's result is not true by giving a hypercyclic operator T such that $T \oplus T$ is not. This yields to ask the analogous question for subspace-hypercyclic operators. In particular, we have

Question 2. If two operators are subspace-hypercyclic, is their direct sum subspace-hypercyclic?

On the other hand, if T satisfies hypercyclic criterion, then $T \oplus T$ is hypercyclic [3]. In 1999, Bés and Peris [4] proved that the converse is also true; i.e, if $T \oplus T$ is hypercyclic, then T satisfies hypercyclic criterion. Now, it is natural to ask analogous questions for subspace-hypercyclic operators. In particular, we have the following questions:

Question 3. If T satisfies subspace-hypercyclic criterion, is $T \oplus T$ subspace-hypercyclic?

Question 4. If $T \oplus T$ is subspace-hypercyclic, does T satisfy hypercyclic criterion?

In the second section of this paper, we answer 1 positively; i.e, if $T_1 \oplus T_2$ is subspace-hypercyclic, then T_1 and T_2 are subspace-hypercyclic. Moreover, we show that 2 has a partial positive answer. In particular, if two operators are subspace-transitive and at least one of them is subspace-mixing, then their direct sum is subspace-transitive which in turn is subspace-hypercyclic. Also, we show that if the direct sum of two operators satisfies subspace-hypercyclic criterion, then both operators do. However, we give counterexamples to show that the converse is not true in general. On the other hand, we show the converse is only true for a single operator; i.e, if T satisfies subspace-hypercyclic criterion, then $T \oplus T$ does, which gives a positive answer to 3. Furthermore, we show that under certain conditions if $T \oplus T$ is subspace-hypercyclic, then T satisfies subspace-hypercyclic criterion which answers 4 partially.

Let \mathcal{M}_1 and \mathcal{M}_2 be subspaces of a Banach space \mathcal{X} , then the direct sum of \mathcal{M}_1 and \mathcal{M}_2 is defined as follows

$$\mathcal{M}_1 \oplus \mathcal{M}_2 = \{(x, y) : x \in \mathcal{M}_1, y \in \mathcal{M}_2\}$$

For more information and details on the direct sum of Banach spaces, the reader may refer to [5]. Let $\{e_i : e_i = (\dots, 0, 1, 0, \dots), i \in \mathbb{Z}\}$ be a standard basis for the sequence space $\ell^2(\mathbb{Z})$, then we have the following theorems.

Theorem 1.1. [2] *Let T be an invertible bilateral forward weighted shift in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n \in \mathbb{Z}}$ and \mathcal{M} be a subspace of $\ell^2(\mathbb{Z})$. Then T is \mathcal{M} -transitive if and only if there exist an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ and $e_{m_i} \in \{e_r\} \cap \mathcal{M}$ such that $T^{n_k} \mathcal{M} \subseteq \mathcal{M}$ for all $k \in \mathbb{N}$ and*

$$\lim_{k \rightarrow \infty} \prod_{j=m_i}^{m_i+n_k-1} w_j = 0 \text{ and } \lim_{k \rightarrow \infty} \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_{-j}} = 0 \quad (1.1)$$

Theorem 1.2. [2] *Let T_1 and T_2 be invertible bilateral forward weighted shifts in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n \in \mathbb{Z}}$ and $\{a_n\}_{n \in \mathbb{Z}}$, respectively. Let \mathcal{M}_1 and \mathcal{M}_2 be closed subspaces of $\ell^2(\mathbb{Z})$. Then $T_1 \oplus T_2$ is $\mathcal{M}_1 \oplus \mathcal{M}_2$ -transitive if and only if there exist $e_{m_i} \in \{e_r\} \cap \mathcal{M}_1$,*

$e_{h_p} \in \{e_r\} \cap \mathcal{M}_2$ and an increasing sequence of positive integers $\{n_k\}_{k \in \mathcal{N}}$ such that $(T_1 \oplus T_2)^{n_k}(\mathcal{M}_1 \oplus \mathcal{M}_2) \subseteq \mathcal{M}_1 \oplus \mathcal{M}_2$ for all $k \in \mathcal{N}$ and

$$\lim_{k \rightarrow \infty} \max \left\{ \prod_{j=m_i}^{m_i+n_k-1} w_j, \prod_{j=h_p}^{h_p+n_k-1} a_j \right\} = 0 \quad (1.2)$$

and

$$\lim_{k \rightarrow \infty} \max \left\{ \prod_{j=1-m_i}^{n_k-m_i} \frac{1}{w_{-j}}, \prod_{j=1-h_p}^{n_k-h_p} \frac{1}{a_{-j}} \right\} = 0 \quad (1.3)$$

2 Main Results

In this paper, all Banach spaces are infinite dimensional separable over the field \mathbb{C} of complex numbers. We always assume that a subspace \mathcal{M} of a Banach space \mathcal{X} is nontrivial ($\mathcal{M} \neq \mathcal{X}$ and $\mathcal{M} \neq \{0\}$) and topologically closed. Also, we assume that every bounded linear operator T is nontrivial ($T \neq I$ and $T \neq 0$, where I is identity operator and 0 is zero operator) unless otherwise stated. We denote $HC(T, \mathcal{M})$ the set of all \mathcal{M} -hypercyclic vectors for T .

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$, and \mathcal{M} be a closed subspace of \mathcal{H} . The following statements are equivalent:*

1. T is \mathcal{M} -transitive,
2. for each $x, y \in \mathcal{M}$, there exist sequences $\{x_k\}_{k \in \mathcal{N}} \subset \mathcal{M}$ and an increasing sequence of positive integers $\{n_k\}_{k \in \mathcal{N}}$ such that $T^{n_k} \mathcal{M} \subseteq \mathcal{M}$ for all $k \geq 1$, $x_k \rightarrow x$ and $T^{n_k} x_k \rightarrow y$ as $k \rightarrow \infty$,
3. for each $x, y \in \mathcal{M}$ and each 0-neighborhood W in \mathcal{M} , there exist $z \in \mathcal{M}$ and $n \in \mathcal{N}$ such that $x - z \in W$, $T^n z - y \in W$ and $T^n \mathcal{M} \subseteq \mathcal{M}$.

Proof. (1) \Rightarrow (2): Let $x, y \in \mathcal{M}$. For all $k \geq 1$, suppose that $B_k = \mathbb{B}(x, 1/k)$ and $B'_k = \mathbb{B}(y, 1/k)$ are two open balls in \mathcal{X} , then $A_k = B_k \cap \mathcal{M}$ and $A'_k = B'_k \cap \mathcal{M}$ are relatively open subsets of \mathcal{M} . By [9, Theorem 3.3.], there exist sequence $\{n_k\}$ in \mathbb{N} and $\{x_k\}$ in \mathcal{M} such that for all $k \geq 1$,

$$x_k \in T^{-n_k}(A'_k) \cap A_k \text{ and } T^{n_k} \mathcal{M} \subseteq \mathcal{M}.$$

It follows that

$$x_k \in A_k \text{ and } T^{n_k}(x_k) \in A'_k.$$

Then, as $k \rightarrow \infty$ the desired result follows.

(2) \Rightarrow (3): Follows immediately from part (2) by taking $z = x_k$ and $n_k = n$ for a large enough $k \in \mathbb{N}$.

(3) \Rightarrow (1): Let U and V be two nonempty open subset of \mathcal{M} . Let W be a neighborhood for zero, pick $x \in U$ and $y \in V$, so there exist $z \in \mathcal{M}$ and $n \in \mathbb{N}$ such that $x - z \in W$, $T^n(z) - y \in W$ and $T^n \mathcal{M} \subseteq \mathcal{M}$. It follows that $U \cap T^{-n}(V) \neq \emptyset$ which proves (1), by [9, Theorem 3.3.]. \square

The following theorem gives a positive answer to 1.

Theorem 2.2. *Let \mathcal{M}_1 and \mathcal{M}_2 be closed subspaces of \mathcal{X} and $T_1 \oplus T_2$ is $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -hypercyclic, then T_1 and T_2 are \mathcal{M}_1 -hypercyclic and \mathcal{M}_2 -hypercyclic, respectively.*

Proof. Let $a \in \mathcal{M}_1$ and $b \in \mathcal{M}_2$, and let $(x, y) \in HC(T_1 \oplus T_2, \mathcal{M}_1 \oplus \mathcal{M}_2)$, then there exist an $\epsilon > 0$ and an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ such that

$$\|(T_1 \oplus T_2)^{n_k}(x, y) - (a, b)\|_{\mathcal{M}_1 \oplus \mathcal{M}_2}^2 \leq \epsilon.$$

It follows that,

$$\|T_1^{n_k}x - a\|_{\mathcal{M}_1}^2 + \|T_2^{n_k}y - b\|_{\mathcal{M}_2}^2 \leq \epsilon.$$

Then

$$\|T_1^{n_k}x - a\|_{\mathcal{M}_1} \leq \epsilon \text{ and } \|T_2^{n_k}y - b\|_{\mathcal{M}_2} \leq \epsilon.$$

Thus, there exists an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ such that $\{T_1^{n_k}x : k \geq 1\}$ and $\{T_2^{n_k}y : k \geq 1\}$ are dense in \mathcal{M}_1 and \mathcal{M}_2 , respectively. Therefore $Orb(T_1, x)$ and $Orb(T_2, y)$ are dense in \mathcal{M}_1 and \mathcal{M}_2 , respectively. \square

The following two results show that the converse of 2.2 holds true under some conditions, which gives a partial positive answer to 2.

Theorem 2.3. *If T_1 and T_2 are \mathcal{M}_1 -transitive and \mathcal{M}_2 -transitive, respectively, and at least one of them is subspace-mixing, then $T_1 \oplus T_2$ is $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -transitive.*

Proof. Suppose that, without loss of generality, that T_1 is \mathcal{M}_1 -mixing. Let $U_1 \oplus V_1$ and $U_2 \oplus V_2$ be open sets in $\mathcal{M}_1 \oplus \mathcal{M}_2$, then U_1 and U_2 are open in \mathcal{M}_1 , and V_1 and V_2 are open in \mathcal{M}_2 . By hypothesis, there exist two numbers $N_1, N_2 \in \mathbb{N}$ such that

$$T_1^{-n}(U_1) \cap U_2 \neq \phi \text{ and } T_1^n(\mathcal{M}_1) \subseteq \mathcal{M}_1 \text{ for all } n \geq N_1$$

and

$$T_2^{-N_2}(V_1) \cap V_2 \neq \phi \text{ and } T_2^{N_2}(\mathcal{M}_2) \subseteq \mathcal{M}_2.$$

Since T_2 is \mathcal{M}_2 -transitive, then $\{n \in \mathbb{N} : T_2^{-n}V_1 \cap V_2 \neq \phi \text{ and } T_2^n\mathcal{M}_2 \subseteq \mathcal{M}_2\}$ is infinite [12]. So, there exists $p \in \mathbb{N}$ such that

$$T_1^{-p}(U_1) \cap U_2 \neq \phi, T_2^{-p}(V_1) \cap V_2 \neq \phi, T_1^p(\mathcal{M}_1) \subseteq \mathcal{M}_1 \text{ and } T_2^p(\mathcal{M}_2) \subseteq \mathcal{M}_2.$$

It follows that

$$(T_1 \oplus T_2)^{-p}(U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \phi \text{ and } (T_1 \oplus T_2)^p(\mathcal{M}_1 \oplus \mathcal{M}_2) \subseteq (\mathcal{M}_1 \oplus \mathcal{M}_2).$$

Thus, $T_1 \oplus T_2$ is $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -transitive. \square

Proposition 2.4. *Let \mathcal{M}_1 and \mathcal{M}_2 be closed subspaces of \mathcal{X} , then T_1 and T_2 are \mathcal{M}_1 -mixing and \mathcal{M}_2 -mixing; respectively, if and only if $T_1 \oplus T_2$ is $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -mixing.*

Proof. For the “if” part, let U_1 and U_2 be open sets in \mathcal{M}_1 , and V_1 and V_2 be open sets in \mathcal{M}_2 , then $U_1 \oplus V_1$ and $U_2 \oplus V_2$ are open in $\mathcal{M}_1 \oplus \mathcal{M}_2$. So there exists an $N \in \mathbb{N}$ such that

$$(T_1 \oplus T_2)^{-n}(U_1 \oplus V_1) \cap (U_2 \oplus V_2) \neq \phi$$

and

$$(T_1 \oplus T_2)^n(\mathcal{M}_1 \oplus \mathcal{M}_2) \subseteq (\mathcal{M}_1 \oplus \mathcal{M}_2)$$

for all $n \geq N$. Then

$$T^{-n}(U_1) \cap U_2 \neq \phi, T^{-n}(V_1) \cap V_2 \neq \phi, T^n(\mathcal{M}_1) \subseteq \mathcal{M}_1 \text{ and } T^n(\mathcal{M}_2) \subseteq \mathcal{M}_2.$$

Therefore, T_1 and T_2 are \mathcal{M}_1 -mixing and \mathcal{M}_2 -mixing, respectively.

The proof of the “only if” part is similar to the proof of 2.3.

□

Proposition 2.5. *If $T_1 \oplus T_2$ satisfies $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -hypercyclic criterion, then T_1 and T_2 satisfy \mathcal{M}_1 -hypercyclic criterion and \mathcal{M}_2 -hypercyclic criterion, respectively.*

Proof. By hypothesis, there exist two dense sets of the form $D_1 \oplus D_2$ and $D_3 \oplus D_4$ in $\mathcal{M}_1 \oplus \mathcal{M}_2$ and an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ such that

- (i) $(T_1 \oplus T_2)^{n_k}(x'_1, x'_2) \rightarrow (0, 0)$ for all $(x'_1, x'_2) \in D_1 \oplus D_2$.
- (ii) for each $(y'_1, y'_2) \in D_3 \oplus D_4$ there exists a sequence $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $(x_k, y_k) \rightarrow 0$ and $(T_1 \oplus T_2)^{n_k}(x_k, y_k) \rightarrow (y'_1, y'_2)$.

Since D_1 and D_3 are dense sets in \mathcal{M}_1 , and D_2 and D_4 are dense sets in \mathcal{M}_2 , then it is easy to show that

- (i) $T_1^{n_k}x'_1 \rightarrow 0$ for all $x'_1 \in D_1$ and $T_2^{n_k}x'_2 \rightarrow 0$ for all $x'_2 \in D_2$,
- (ii) for each $y'_1 \in D_3$ there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \mathcal{M}_1$ such that $x_k \rightarrow 0$ and $T_1^{n_k}x_k \rightarrow y'_1$, and for each $y'_2 \in D_4$ there exists a sequence $\{y_k\}_{k \in \mathbb{N}} \subset \mathcal{M}_2$ such that $y_k \rightarrow 0$ and $T_2^{n_k}y_k \rightarrow y'_2$.

it is readily seen that T_1 and T_2 satisfy \mathcal{M}_1 -hypercyclic criterion and \mathcal{M}_2 -hypercyclic criterion, respectively.

□

The following examples show that the converse of 2.5 is not true in general.

Example 2.6. Let \mathcal{M}_1 and \mathcal{M}_2 be two subspaces of \mathcal{X} such that $\mathcal{M}_1 \oplus \mathcal{M}_2 \neq \mathcal{X}$. If for any two increasing sequences of positive integers $\{n_k\}_{k \in \mathbb{N}}$ and $\{m_k\}_{k \in \mathbb{N}}$, we have $\{n_k\}_{k \in \mathbb{N}} \cap \{m_k\}_{k \in \mathbb{N}} = \emptyset$ whenever T_1 satisfies \mathcal{M}_1 -hypercyclic criterion with respect to $\{n_k\}_{k \in \mathbb{N}}$ and T_2 satisfies \mathcal{M}_2 -hypercyclic criterion with respect to $\{m_k\}_{k \in \mathbb{N}}$. Then, $T_1 \oplus T_2$ does not satisfy $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -hypercyclic criterion.

Example 2.7. Let \mathcal{M}_1 and \mathcal{M}_2 be closed subspaces of the Banach space $\ell^p(\mathbb{Z})$. Choose sequences $\{w_n\}_{n \in \mathbb{Z}}$ and $\{a_n\}_{n \in \mathbb{Z}}$ of positive real numbers such that each one satisfies 1.1 but not either Equation (1.2) or Equation (1.3) of 1.2. Then, suppose that T_1 and T_2 are bilateral weighted shifts with weight sequence $\{w_n\}_{n \in \mathbb{Z}}$ and $\{a_n\}_{n \in \mathbb{Z}}$, respectively. It follows from 1.1 and 1.2 that T_1 and T_2 satisfy \mathcal{M}_1 -hypercyclic criterion and \mathcal{M}_2 -hypercyclic criterion, respectively. However, $T_1 \oplus T_2$ does not satisfy $(\mathcal{M}_1 \oplus \mathcal{M}_2)$ -hypercyclic criterion.

The following proposition shows that the converse of 2.5 holds true for a single operator T which gives a positive answer to 3.

Proposition 2.8. *If T satisfies \mathcal{M} -hypercyclic criterion, then $T \oplus T$ satisfies $(\mathcal{M} \oplus \mathcal{M})$ -hypercyclic criterion.*

Proof. Since T satisfies \mathcal{M} -hypercyclic criterion, then there exist two dense subsets D_1 and D_2 of \mathcal{M} and an increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ such that all hypothesis of subspace-hypercyclic criterion are satisfied. It is easy to show that $D_1 \oplus D_1$ and $D_2 \oplus D_2$ are dense sets in $\mathcal{M} \oplus \mathcal{M}$. Let $(x_1, x_2) \in D_1 \oplus D_1$ then $x_1, x_2 \in D_1$. By hypothesis, $T^{n_k}x_1 \rightarrow 0$ and $T^{n_k}x_2 \rightarrow 0$. Thus,

$$(T \oplus T)^{n_k}(x_1, x_2) \rightarrow (0, 0). \quad (2.1)$$

Now, let $(y_1, y_2) \in D_2 \oplus D_2$ then $y_1, y_2 \in D_2$. Also, by hypothesis, there exist two sequences say $\{x_k\}_{k \in \mathbb{N}}$ and $\{y_k\}_{k \in \mathbb{N}}$ in \mathcal{M} such that

$$x_k \rightarrow 0 \text{ and } T^{n_k}x_k \rightarrow y_1,$$

and

$$y_k \rightarrow 0 \text{ and } T^{n_k}y_k \rightarrow y_2.$$

Therefore,

$$(x_k, y_k) \rightarrow (0, 0) \text{ and } (T \oplus T)^{n_k}(x_k, y_k) \rightarrow (y_1, y_2). \quad (2.2)$$

Finally, since $T^{n_k}\mathcal{M} \subseteq \mathcal{M}$, then

$$(T \oplus T)^{n_k}(\mathcal{M} \oplus \mathcal{M}) \subseteq \mathcal{M} \oplus \mathcal{M}. \quad (2.3)$$

It follows by Equation (2.1), Equation (2.2) and Equation (2.3) that $T \oplus T$ satisfies $\mathcal{M} \oplus \mathcal{M}$ -hypercyclic criterion, and thus hypercyclic. \square

Corollary 2.9. *If T satisfies subspace-hypercyclic criterion, then $T \oplus T$ is subspace-hypercyclic.*

For hypercyclicity, Bés and Peris [4] proved that the converse of the 2.9 above is also true. The next theorem shows that the converse of 2.9 is true under certain conditions, which gives a partial positive answer to 4. First we need the following lemma.

Lemma 2.10. *Let $T, S \in \mathcal{B}(\mathcal{X})$ satisfy the equation $TS = ST$ and \mathcal{M} be a subspace of \mathcal{X} . If $x \in HC(T, \mathcal{M})$, then $Sx \in HC(T, S\mathcal{M})$.*

Proof. Let $x \in HC(T, \mathcal{M})$, then $Orb(T, x) \cap \mathcal{M}$ is dense in \mathcal{M} . Thus

$$\begin{aligned} \overline{Orb(T, Sx) \cap S\mathcal{M}} &= \overline{\{T^n Sx : n \geq 0\} \cap S\mathcal{M}} \\ &= \overline{\{ST^n x : n \geq 0\} \cap S\mathcal{M}} \\ &= \overline{S \{T^n x : n \geq 0\} \cap S\mathcal{M}} \\ &\supseteq \overline{S(\{T^n x : n \geq 0\} \cap \mathcal{M})} \\ &\supseteq S(\overline{\{T^n x : n \geq 0\} \cap \mathcal{M}}) \\ &= S\mathcal{M}. \end{aligned}$$

It follows that $Orb(T, Sx) \cap S\mathcal{M}$ is dense in $S\mathcal{M}$. Thus T is $S\mathcal{M}$ -hypercyclic operator with subspace-hypercyclic vector Sx . \square

Theorem 2.11. *Let $T \oplus T$ be $(\mathcal{M} \oplus \mathcal{M})$ -hypercyclic. If there exists a subspace-hypercyclic vector (x, y) such that x has the following properties*

$$(i) \ x \in \bigcap_{n \in \mathbb{N}} T^n(\mathcal{M}),$$

(ii) $T^j \mathcal{M} \subseteq \mathcal{M}$ whenever $T^j x \in \text{Orb}(T, x) \cap \mathcal{M}$.

Then T satisfies \mathcal{M} -hypercyclic criterion.

Proof. Let $(x, y) \in HC(T \oplus T, \mathcal{M} \oplus \mathcal{M})$ such that x satisfies the stated conditions. Since $x \in HC(T, \mathcal{M})$ (by 2.2), we show that the \mathcal{M} -hypercyclicity criterion is satisfied by the dense sets $D = D_1 = D_2 = \text{Orb}(T, x) \cap \mathcal{M}$. Since the operator $I \oplus T^n$ commute with $T \oplus T$, then by 2.10, we have

$$(x, T^n y) \in HC(T \oplus T, \mathcal{M} \oplus T^n \mathcal{M}) \quad (2.4)$$

for all $n \in \mathbb{N}$. Since $y \in HC(T, \mathcal{M})$ (by 2.2), then $\text{Orb}(T, y) \cap \mathcal{M}$ intersects every open set in \mathcal{M} . Let U be an open neighborhood of 0 in \mathcal{M} , then there exist $u_0 \in U \subseteq \mathcal{M}, r \in \mathbb{N}_0$ such that $u_0 = T^r y$. It follows by Equation (2.4) that

$$(x, u_0) \in HC(T \oplus T, \mathcal{M} \oplus T^r \mathcal{M}).$$

By condition i, we have $(0, x) \in \mathcal{M} \oplus T^r \mathcal{M}$, then there exists $n_0 \in \mathbb{N}$ such that

$$(T^{n_0} x, T^{n_0} u_0) \in \text{Orb}(T \oplus T, (x, u_0)) \cap (\mathcal{M} \oplus T^r \mathcal{M})$$

is arbitrarily close to $(0, x)$. Since $T^{n_0} x \in \mathcal{M}$, then $T^{n_0} x \in D$, and so $T^{n_0} \mathcal{M} \subseteq \mathcal{M}$ by condition ii. By continuing the same process, we get a sequence $u_k \rightarrow 0$ in \mathcal{M} and $n_k \in \mathbb{N}$ such that $T^{n_k} x \rightarrow 0, T^{n_k} u_k \rightarrow x$ and $T^{n_k} \mathcal{M} \subseteq \mathcal{M}$.

Let $C = \{i \in \mathbb{N}_0 : T^i x \in D\}$, and let us define maps $S_{n_k} : D \rightarrow \mathcal{M}$ by $S_{n_k}(T^i x) = T^i u_k; i \in C$ (by condition ii, it is clear that $S_{n_k}(T^i x) \in \mathcal{M}$ for all $k \in \mathbb{N}_0, i \in C$). Now, for all $T^i x \in D$, we have

$$\begin{aligned} T^{n_k}(T^i x) &= T^i(T^{n_k} x) \rightarrow 0, \\ S_{n_k}(T^i x) &= T^i u_k \rightarrow 0 \text{ since } u_k \rightarrow 0, \\ T^{n_k} S_{n_k}(T^i x) &= T^i T^{n_k} u_k \rightarrow T^i x. \end{aligned}$$

for all $i \in C$. It follows that T satisfies \mathcal{M} -hypercyclic criterion. □

It follows by 2.11 above that hypercyclicity and subspace-hypercyclicity do not share the same properties. Therefore, one may wonder whether there are some more similarities and differences between hypercyclicity and subspace-hypercyclicity.

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